Int. J. Solids Structures, 1974, Vol. 10, pp. 985–991. Pergamon Press. Printed in Gt. Britain.

ON GENERAL CONSERVATIVE LOADING OF NATURALLY CURVED AND TWISTED FUNICULAR RODS

MEHDI FARSHAD
Pahlavi University, Shiraz, Iran

(*Received* 2 *October* 1973; *revised* 31 *January* 1974)

Abstract—Considered is the problem of a naturally curved and pretwisted rod subjected to general motion-dependent end and lateral loading. The bar is assumed to be a spatial funicular curve in its reference state of equilibrium. Through the utilization of kinetic stability considerations, and the notions of self-adjoint systems, the conditions under which the problem can be considered conservative are obtained. The results are applied to some cases of technical importance.

1. INTRODUCTION

Among the various results obtained through the extensive investigations related to the stability of deformable media several deductions appear to be of paramount importance. It has been established that the loss of elastic stability of equilibrium is in general either by divergence (bifurcation of equilibrium) or flutter (vibrations with increasing amplitude). It is also known that the modes of instability as well as the methods of stability analyis are highly dependent on the nature of the loading state which may, in turn, be put in the general classification of conservative and nonconservative loading systems. Accordingly the classification of methods of stability analysis into static, kinetic, imperfection and dynamic approaches has been made (see, e.g. [1–6]). Moreover, it has been observed that the mathematical treatment of nonconservative problems would usually lead to non-selfadjoint governing equations. To above deductions one may add the note that most of work done in this area has been concentrated on the stability problems wherein the reference states the stability of which were to be studied could be taken to be the undeformed state of the body: this situation would in fact justify the extensively used linear methods of stability analysis.

Based on the forgoing citations and due to importance of the nature of applied loading in the stability outcome attention has been paid to the end of load categorization and solution of problems for each types of loads. From the mathematical point of view there exists a close link between the notions of adjoint mathematical systems and conservative and/or nonconservative physical problems. In this direction, some attempts have been made that utilize the concepts of adjoint systems based on which certain conservative problems of bars could be found [7, 8]. Moreover, using the kinetic approach of stability analysis together with the notion of adjoint systems the general conservative end-loading of pretwisted rods was obtained[9]. The same line of approach was taken in characterization of general conservative end-loading of bars in connection with an optimization problem[10] and was generalized in the characterization of general conservative loading of funicular arches[11].

The main goal of the present paper is the characterization of the domains of conservative and non-conservative loading states of a pretwisted curved bar which in the reference state of equilibrium is a spatial funicular arch. To this end the kinetic approach of stability analysis is adopted and through a systematic procedure the conditions which render the governing equations self-adjoint (and hence the system conservative) are obtained. Several illustrative cases are also considered. As it was pointed out in[9], such a systematic procedure of identifying the classes of conservative and nonconservative systems does not seem to be restricted to bar problems and may be fruitfully applicable to other systems under various loading conditions. For related works on the subject of pretwisted curved rods, (see, e.g.[12–18]).

2. BASIC EQUATIONS

Consider a spatial rod which has an arbitrary cross-section and a general three-dimensional curve as its undeformed centerline. Let $X_o Y_o Z_o$ be a global fixed set of coordinate axes and let $\tau_o n_o b_o$ be a coordinate system defined at each point of the undeformed curve in such a way that τ_o, n_o, b_o, in respective fashion, are directed along the tangent, principal normal, and binomial to the curve. In addition, we also define a third coordinate system $x_o y_o z_o$ at each point of undeformed centerline with x_o and y_o taken along the principal directions of the cross-section and z_o along the tangent to the curve. The position of $\tau_o n_o b_o$ axes, relative to $x_o y_o z_o$ system, is obtained completely if the arc length parameter, s and the rate of change of orientation of $\tau_o n_o b_o$, relative to $x_o y_o z_o$, is specified. If p_o, q_o denote the components of curvature and if r_o is the rate of twist of the curve (all in undeformed state of bar), then representing the unit vectors in $x_o y_o z_o$ by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ symbols we may express such a vector by

$$\boldsymbol{\omega} = p_o\mathbf{i} + q_o\mathbf{j} + r_o\mathbf{k}.$$

In an arbitary state of deformed rod the axes of coordinate systems $\tau_o n_o b_o$ and $x_o y_o z_o$ all occupy new positions τnb and xyz, respectively. If u, v, w represent the components of displacement vector of the origin of xyz relative to $x_o y_o z_o$ and if ϕ, ψ, χ denote the angle-between the corresponding axes then the general state of deformation of any point of the bar is representable by the specification of six quantities u, v, w, ϕ, ψ and χ. Furthermore, let V_x, V_y, V_z represent the components of force resultant and let M_x, M_y, M_z be the components of the couple resultant all relative to xyz system. With these priliminary considerations the governing equations of elastic deformation of bar can be written as follows: (see, e.g.[2]).

Force equations of motion:

$$V_x' + qV_z - rV_y + F_x = 0 \tag{1}$$

$$V_y' + rV_x - pV_z + F_y = 0 \tag{2}$$

$$V_z' + pV_y - qV_x + F_z = 0 \tag{3}$$

$$M_x' + qM_z - rM_y - V_y = 0 \tag{4}$$

$$M_y' + rM_x - pM_z + V_x = 0 \tag{5}$$

$$M_z' + pM_y - qM_x = 0. \tag{6}$$

Constitutive relations:

$$\begin{aligned} M_x &= \alpha(\phi' + q_o\chi - r_o\psi) \\ M_y &= \beta(\psi' + r_o\phi - p_o\chi) \\ M_z &= \gamma(\chi' + p_o\psi - q_o\phi). \end{aligned} \tag{7}$$

Kinematical relations:

$$\psi = u' + q_o w - r_o v \tag{8}$$

$$-\phi = v' + r_o u - p_o w \tag{9}$$

$$0 = w' + p_o v - q_o u. \tag{10}$$

In the above equations the prime symbol denotes the partial derivative with respect to S, F_x, F_y, F_z are the components of the external and/or inertia forces, α, β, γ are the bending and torsional rigidity of the bar, and p, q, r are the curvature and twist in the deformed state. It is to be noted that in writing the above governing equations we have neglected the effects of shear deformation and warping of the cross sections. The bar centerline has also been assumed to be inextensible.

If the mass per unit length of bar is denoted by m and t denotes the time parameter, the general form of F_x, F_y, F_z can be written as:

$$\begin{aligned} F_x &= -m\frac{\partial^2 u}{\partial t^2} + f_x \\ F_y &= -m\frac{\partial^2 y}{\partial t^2} + f_y \\ F_z &= -m\frac{\partial^2 \omega}{\partial t^2} + f_z. \end{aligned} \tag{11}$$

f_x, f_y, f_z are the components of the applied loads which in this paper are taken to have a general character; they may be motion dependent as well as dead loads. If $\mathbf{r}_o$ represents the position vector of undeformed centerline and if $\mathbf{r}$ is the position vector in the deformed state such that:

$$\mathbf{r} = \mathbf{r}_o + u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \tag{12}$$

then we have for general type of loading:

$$\mathbf{f} = (f_x, f_y, f_z) = \mathbf{f}(\mathbf{r}_o, \mathbf{r}, s, t) \tag{13}$$

more explicit form of motion dependency of **f** and its relation to the conservativeness and/or nonconservativeness of our system shall be discussed in a latter section.

3. SPATIAL FUNICULAR BARS

A general spatial curved and pretwisted rod, with a prescribed centerline curve, is said to be a funicular curve for a certain state of loading if under that loading there exists a state of internal axial force in the bar and all the bending moments and shear forces are zero. Conversely, for a given loading one can always find the geometry of centerline curve which represents a funicular curve for that loading. If the additional assumption of inextensibility is imposed on the funicular bar then one may regard the funicular state as a stressed state with no deformation. In this paper we shall choose a linear theory of dynamic stability analysis which will render itself to the complete characterization of the types of conservative as well as non-conservative loading. The use of such a linear theory is justified only if the initial state of system whose stability is under consideration does not involve any deformation prior to unstable state. The spatial curved bar with the above-mentioned funicular properties satisfies this requirement and hence our usage of a linear theory of dynamic stability investigation is completely justified. Mathematically, the governing equations of spatial funicular curves are deduced from equations (1)–(6) by imposing the demand that all force results except $T_o(V_z$ at funicular state) vanish; if we do so we obtain.

$$q_o T_o + f_x^o = 0 \tag{14}$$

$$-p_o T_o + f_y^o = 0 \tag{15}$$

$$T_o' + f_z^o = 0 \tag{16}$$

wherein

$$\mathbf{f}^o = (f_x^o, f_y^o, f_z^o) = \mathbf{f}\Big|_{\mathbf{r}=\mathbf{r}_o}. \tag{17}$$

It is to be pointed out that the above equations as well as the other results of this paper will reduce to the more specialized case of plane funicular arch if the torsion r_o is set equal to zero. Equation (14)–(17) represent the initial state of our system.

4. STABILITY ANALYSIS OF SPATIAL FUNICULAR ARCHES

To the end of determining the class of conservative and/or nonconservative loading states to which a spatial rod may be subjected we make use of the notions of kinetic approach of stability analysis. A typical linear dynamic stability analysis of the system is achieved through the introduction of some perturbation in the system under study and the investigation of ensuing motion about the reference state. In the present problem the reference state is characterized by the position vector $\mathbf{r}_o$ and the corresponding governing equations are give by (14)–(17). The perturbed state is represented by the position vector $\mathbf{r}$ as defined by (12). In a general perturbed state of stress and deformation the bar is no longer funicular and there exist shear forces as well as bending moments at each point of the centerline. Equations (1)–(10) represent the governing equations of bar in any linear perturbed state, provided M_x, M_y, M_z and V_x, V_y are envisaged to be the perturbations in the force quantities (which were all zero in the funicular state) and noting that:

$$\bar{V}_z = T_o + V_z. \tag{18}$$

At this stage of stability analysis the form of applied loading must be specified. We consider a loading state with a fairly general character. The motion dependency of loads are specified by the following expansions:

$$(f_x, f_y, f_z) = (f_x^o, f_y^o, f_z^o) + (\mathbf{f}^{(1)}, \mathbf{f}^{(1)}, \mathbf{f}^{(1)}) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) + (\mathbf{f}_x^{(2)}, \mathbf{f}_y^{(2)}, \mathbf{f}_z^{(2)}) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k})' \tag{19}$$

let

$$\begin{aligned} \mathbf{f}_x^{(1)} &= f_{11}\mathbf{i} + f_{12}\mathbf{j} + f_{13}\mathbf{k} \\ \mathbf{f}_x^{(2)} &= f_{21}\mathbf{i} + f_{22}\mathbf{j} + f_{23}\mathbf{k}. \end{aligned} \tag{20}$$

Similarly

$$\begin{aligned} \mathbf{f}_y^{(1)} &= g_{11}\mathbf{i} + g_{12}\mathbf{j} + g_{13}\mathbf{k} \\ \mathbf{f}_y^{(2)} &= g_{21}\mathbf{i} + g_{22}\mathbf{j} + g_{23}\mathbf{k} \end{aligned} \tag{21}$$

and

$$\begin{aligned} \mathbf{f}_z^{(1)} &= t_{11}\mathbf{i} + t_{12}\mathbf{j} + t_{13}\mathbf{k} \\ \mathbf{f}_z^{(2)} &= t_{21}\mathbf{i} + t_{22}\mathbf{j} + t_{23}\mathbf{k}. \end{aligned} \tag{22}$$

Now, using the Fresnet–Seret formula,

$$\begin{aligned} \mathbf{i}' &= \mathbf{r}_o \mathbf{j} - q_o \mathbf{k} \\ \mathbf{j}' &= -r_o \mathbf{i} + p_o \mathbf{k} \\ \mathbf{k}' &= q_o \mathbf{i} - p_o \mathbf{j} \end{aligned} \tag{23}$$

and with the help of relations (8)–(10) and (20)–(22) expressions (19) become

$$\begin{aligned} f_x &= f_x^o + f_{11} u + f_{12} v + f_{13} w + f_{21} \psi - f_{22} \phi \\ f_y &= f_y^o + g_{11} u + g_{12} v + g_{13} w + g_{21} \psi - g_{22} \phi \\ f_z &= f_z^o + t_{11} u + t_{12} v + t_{13} w + t_{21} \psi - t_{22} \phi. \end{aligned} \tag{24}$$

The applied loads expressed by relations (24) embody the special cases of technical importance such as dead loadings and follower type of loadings.

To perform the stability analysis by kinetic methods we use the classical approach of assuming a time harmonic solution for any of the quantities involved. So if $Q(s, t)$ typifies such a quantity we write:

$$Q(s, t) = Q(s) e^{i\zeta t} \tag{25}$$

where ζ is the frequency of oscillatory motions. It is well known (see, e.g.[2]) that the study of the dynamic stability problem is usually centered around the investigation of load frequency relation. As a particular point of interest one usually observes that the change of value of frequency from a real quantity to a complex one, which is brought about by its passage through zero, usually determines the nature of the modes of instability (divergence or flutter). On the other hand viewing the frequency as the eigenvalue of the problem and referring to the mathematical theory of self adjoint and non-self adjoint systems we conclude that by studying the conditions for the real-valuedness of the eigenvalues of the eigenproblem we may be able to make some deductions regarding the system as being conservative (with divergence as the mode of loss of stability) or a nonconservative one (with flutter as the mode of loss of stability). To this end if we use a form of solution similar to (25) and use it in equations (1)–(6) and, with the help of (11) and (24) we can define three linear operators in terms of which the equations become,

$$\begin{aligned} L_1 &\equiv V_x' + q_o V_z - r_o V_y + (\psi' + r_o \phi) T_o + f_x - f_x^o = m\zeta^2 u \\ L_2 &\equiv V_y' + r_o V_x - p_o V_z - (\phi' - r_o \psi) T_o + f_y - f_y^o = m\zeta^2 v \\ L_3 &\equiv V_z' + p_o V_y - q_o V_x + f_z - f_z^o = m\zeta^2 w. \end{aligned} \tag{26}$$

5. CONSERVATIVELY LOADED SPATIAL FUNICULAR ARCHES

To investigate the self-adjointness of system following a scheme similar to one used in [9–11], we form the concommittant of the system, we have,

$$\Delta = (\zeta^2 - \bar{\zeta}^2) \int_0^L (u\bar{u} + v\bar{v} + w\bar{w})\, ds = \int_0^L [(\bar{u} L_1 + \bar{v} L_2 + \bar{w} L_3) - (u \bar{L}_1 + v \bar{L}_2 + w \bar{L}_3)]\, ds \tag{27}$$

wherein L is the total length of the curved rod and the bar over a letter signifies that the letter with the bar is the complex conjugate of the corresponding letter without the bar. If we utilize the expressions for L_1, L_2, L_3 from (26) in (27) and with the help of equations (1)–(10) and after a fairly lengthy manipulation of symbols we obtain,

$$\begin{aligned}\Delta = [(\bar{u}V_x - u\bar{V}_x) + (\bar{u}V_y - v\bar{V}_y) + (\bar{w}V_z - w\bar{V}_z) + (\bar{\psi}M_y - \psi\bar{M}_y) + (\bar{\phi}M_x - \phi\bar{M}_x) \\ + T_o(\bar{u}u' + \bar{v}v' - 2r_o\bar{u}v - 2r_o u\bar{v} - u\bar{u}' - v\bar{v}') + t_{21}(\bar{w}u - w\bar{u}) + t_{22}(\bar{w}v - w\bar{v})]_o^L \\ + \int_o^L [(u\bar{v} - \bar{u}v)(f_{12} - g_{11} + r_o f_{21} - 2r_o T_o' + r_o g_{22} + q_o t_{22} + p_o t_{21}) \\ + (\bar{u}w - u\bar{w})(f_{13} - t_{11} + q_o f_{21} - q_o T_o' - p_o f_{22} + r_o p_o T_o - r_o t_{22} + t_{21}' - f_x^{o\prime}) \\ + (\bar{v}w - v\bar{w})(g_{13} - t_{12} + q_o g_{21} + r_o q_o T_o' - p_o g_{22} + p_o T_o' + r_o t_{21} + t_{22}' - f_y^{o\prime}) \\ + (\bar{u}u' - u\bar{u}')(f_{21} - T_0') + (\bar{v}v' - v\bar{v}')(g_{22} - T_0') \\ + f_{22}(\bar{u}v' - u\bar{v}') + g_{21}(\bar{v}u' - v\bar{u}')]\, ds. \end{aligned} \quad (28)$$

Now, if we require that the system of equations (20) be self adjoint, to fulfill such a requirement the concommittant Δ must vanish. Hence for a general state of deformation and for the general boundary conditions we conclude from (28) that the following conditions must be satisfied.

$$\begin{aligned}[(\bar{u}V_x - u\bar{V}_x) + (\bar{v}V_y - v\bar{V}_y) + (\bar{w}V_x - w\bar{V}_x) + (\bar{\psi}M_y - \psi\bar{M}_y) + (\bar{\rho}M_x - \rho\bar{M}_x) \\ + T_o(\bar{u}u' + \bar{v}v' - 2r_o\bar{u}v - 2r_o u\bar{v} - u\bar{u}' - v\bar{v}') + t_{21}(\bar{w}u - w\bar{u}) + t_{22}(\bar{w}v - w\bar{v})]_o^L = 0\end{aligned} \quad (29)$$

$$\begin{aligned} & f_{12} - g_{11} + r_o f_{21} - 2r_o T_o' + r_o g_{22} + q_o t_{22} + p_o t_{21} = 0 \\ & f_{13} - t_{11} + q_o f_{21} - q_o T_o' - p_o f_{22} + r_o p_o T_o - r_o t_{22} + t_{21}' - f_x^{o\prime} = 0 \\ & g_{13} - t_{12} + q_o g_{21} + r_o q_o T_o - p_o g_{22} + p_o T_o' + r_o t_{21} + t_{22}' - f_y^{o\prime} = 0 \\ & f_{21} - T_o' = 0 \\ & f_{22} = 0 \\ & g_{22} - T_o' = 0 \\ & g_{21} = 0. \end{aligned} \quad (30)$$

Relations (29) and (30) constitute the totality of conditions which are to be satisfied if the mathematical system is to be self-adjoint. As we pointed out such a situation can be directly linked to the realization of a conservative system. Accordingly, if these conditions are fulfilled the system is conservative and if not the system is nonconservative. We note that part of these requirements are related to the state of loading along the curved bar itself while the others are the boundary requirements. We shall now proceed to apply these general results to some cases of importance.

5. EXAMPLES

5.1 *Plane funicular arches*

For a plane arch $r_o = 0$, and we can arbitrarily choose $q_o = 0$. Correspondingly we may set $u \equiv 0$. Considering first the case of dead loading, $f_x^o = 0$, $f_y^o = 0$, $f_z^o \neq 0$ and assuming that conditions (29) are satisfied by some appropriate boundary conditions we can verify that conditions (30) are also satisfied provided,

$$f_{21} = T_o', \quad g_{22} = T_o'$$

which corresponded to the state of dead loading; these specific results were also obtained in [11]. As a second case of interest we can easily verify that the plane circular ring subjected to constant hydrostatic pressure (follower load) is also conservative.

5.2 *Follower type loading of helical bars*

As another technically important case of spatial curves consider a helical bar with some boundary conditions satisfying (29). For a helical curve the quantities r_0, p_0, q_0 are constant. Suppose that the circular helical arch is acted upon by the forces which remain normal to the curve so that $f_z^o = 0$. Hence, the funicular curve equations (14)–(16) yield,

$$T_0 = \text{constant}$$
$$f_x^o = -q_0 T_0, \qquad f_y^o = p_0 T_0 .$$

If this state of follower type loading is substituted into conditions (30) the conclusion is reached that due to presence of r_0 dependent terms those conditions will not be satisfied unless $r_0 = 0$ which would be the case of a plane circular ring under hydrostatic pressure. So unlike the case of a circular ring which is conservative under hydrostatic pressure, a helical arch under hydrostatic pressure is nonconservative.

REFERENCES

1. H. Ziegler, On the concept of elastic stability, in *Advances in Applied Mechanics*. Academic Press, New York (1956).
2. V. V. Bolotin, *Nonconservative Problems of the Theory of Elastic Stability*. Pergamon Press, Oxford (1963).
3. G. Herrmann, Stability of equilibrium of elastic systems subjected to nonconservative forces, *Appl. Mech. Rev.* **20** (2), 103–108 (1967).
4. H. Ziegler, *Principles of Structural Stability*. Blaisdel, London (1968).
5. H. Leipholz (Editor), Instability of continuous systems, *IUTAM Symposium* Herrenalb (1969). Springer, Berlin (1971).
6. H. Leipholz, *Stability Theory*. Academic Press, New York (1970).
7. S. Nemat-Nasser and G. Herrmann, Adjoint systems in non-conservative problems of elastic stability, *AIAA J.* **4**, 2221–2222 (1966).
8. S. Nemat-Nasser, On circulatory force fields and their adjoints. *Trans. ASME*, Ser. E, *J. Appl. Mech.* **90**, 829–831 (1968).
9. M. Farshad, On general conservative end loading of pretwisted rods, *Int. J. Solids Struct.* **9**, 1361–1371 (1973).
10. M. Farshad and I. Tadjbakhsh, Optimum shape of columns with general conservative end loading, *J. Optimization Theory Applications* **11** (4), 413–420 (1973).
11. I. Tadjbakhsh and M. Farshad, On conservatively loaded funicular arches and their optimal design. *IUTAM Symp. on Structural Optimization* (1973).
12. H. Leipholz, Ein Beitrag Zü Dem Problem Der Knickung Einer Geraden Welle Durch Druck Ünter Torsionsbeauspruchung, Dissertation, Library of University of Stuttgart, West-Germany.
13. E. Hui, Knickung Verundener Stäbe Unter Druck Österr, *Ing. Archiv.* **9**, 288–319 (1955).
14. E. P. Popov, *Nichtlineare Probleme der Statik Dunner Stäbe.* GITTL Leningrad, Moskow (1948).
15. H. Ziegler, Dir Knickung der Verwundensen Stäbes, *Schueizer Bauztg.* **66** (1948).
16. H. Ziegler, Stabilitatsprobleme bei geraden Stäben ü. Wellen, *ZAMP* **2**, 265–289 (1951).
17. H. Ziegler, Knickung Gerader Stabe unter Torsion, *ZAMP* **3**, 96–119 (1952).
18. A. Trösch, Stabilitätsprobleme bei Tordierten Wellen ü. Staben *Ing. Arch.* **20**, 258–277 (1952).

Абстракт — Исследуется задача естественно искрывленного и предварительно закрученного стержня, подверженного действию общей, зависящей от движения края и поперечной нагрузки. Предполагается, что стержень является пространственный канатной кривой для своего исходного состояния равновесия. Путем использования обсуждений кинетической устойчивости и понятий самосопряженных систем получаются условия, для которых можно рассматривать задачу в смысле консервативной. Применяются результаты для некоторых случаев проявляющих техническое значение.